Large-N Expansion for a Nucleon-Nucleon Potential

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The Schrödinger equation has been solved by $1/N$ expansion for a two nucleon system which interacts by an attractive Yukawa potential. For the ground and first excited states, energy eigenvalues have been obtained.

The Schrödinger equation of quantum mechanical systems can be solved by large-N expansions [1–4]. We have applied this method to obtain the energies of the ground and first excited states of the deuteron. The calculations are carried out by simple algebraic recursion methods.

The radial part of the N-dimensional Schrödinger equation has the form

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)+\frac{l(l+N-2)}{2mr^2}+V_N(r)\right]R_{nl}(r) = E_n R_{nl}(r), \tag{1}$$

where $l(l+N-2)$ is the eigenvalue of the generalised angular momentum operator and m the reduced mass.

For the deuteron, m is half of its mass and the Yukawa type interaction potential in N dimensions takes the form

$$v_N(r) = -v_0 \exp\{-9\alpha r/N^2\}/r. \tag{2}$$

From (1) and (2), putting $R(r)=\exp\{-(N-1)/2\}\,u(r)$ and giving lengths in terms of the π-meson wave length ($1/\lambda\!\!\!^{-}_\pi=\mu_0=0.7\ F^{-1}$) and energies in terms of $\varepsilon=2\mu_0^2\hbar^2/m=40.64$ MeV, one obtains

$$-\frac{\mathrm{d}^2}{2\,\mathrm{d}r^2}u(r)+k^2\left[\frac{1}{8r^2}\left(1-\frac{1}{k}\right)\left(1-\frac{3}{k}\right) - \tilde{a}\,\frac{\exp\{-9\alpha r/N^2\}}{r}\right]u(r) = E\,u(r), \tag{3}$$

where $\tilde{a}=V_0/k^2$ and $k=N+2l$.

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Ground State ($l=0$)

The most important term ($N\to\infty$) is

$$E_\infty = k^2 E^{(-2)} = k^2\left[\frac{1}{8r_0^2}-\frac{\tilde{a}}{r_0}\right], \tag{4}$$

where r_0 is obtained by minimization of the potential $\frac{1}{8r^2}-\frac{\tilde{a}}{r}$;

$$r_0 = 1/4\tilde{a}, \quad E^{(-2)} = -2\tilde{a}^2 \tag{5}$$

In order to make higher order corrections to the ground state energy, one writes $x=r(x)-r_0$ and chooses the ground state wave function as

$$u_0(r) = e^{\Phi_0(x)}. \tag{6}$$

Then (3) transforms to the Riccati equation

$$-\tfrac{1}{2}(\Phi_0''(x)+\Phi_0'^2(x))+k^2V_{\mathrm{eff}}(x) +(-\tfrac{1}{2}k+\tfrac{3}{8})\,r^{-2}(x)+Q(x)=\mathscr{E}_0, \tag{7}$$

where

$$V_{\mathrm{eff}}(x) = \frac{1}{8r^2(x)}-\frac{\tilde{a}}{r(x)}+2\tilde{a}^2 \tag{8}$$

and V_{eff} is chosen in such a way that $V_{\mathrm{eff}}(0)=0$. In (7), $\mathscr{E}_0$ and $Q(x)$ are given by

$$\mathscr{E}_0 = E_0 - k^2 E^{(-2)}, \tag{9}$$

$$Q(x) = 9\alpha\tilde{a}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\left(\frac{9\alpha}{k^2}r(x)\right)^2, \tag{10}$$

where E_0 is the ground state energy.

The recurrence relations for $E_0^{(n)}$ and $\Phi_0^{(n)}$ are obtained by substituting the expansions

$$\mathscr{E}_0 = \sum_{n=-1}^{\infty}E_0^{(n)}k^{-n}, \tag{11}$$

$$\Phi_0'(x) = \sum_{n=-1}^{\infty}\Phi_0^{(n)}(x)\,k^{-n} \tag{12}$$

and $Q(x)$ of (10) into (9) and combining terms of the same order in k. $E_0^{(n)}$ are determined from the solution at $x=0$.

From (9) and (11) the ground state energy is calculated to be

$$E_0 = k^2E^{(-2)}+\sum_{n=-1}^{\infty}k^{-n}E_0^{(n)}. \tag{13}$$

When $E_0^{(n)}$ is substituted in (13) ($k=3$ is taken) the ground state energy in units of MeV becomes

$$E_0 = \frac{1}{\varepsilon}V_0^2\,S_0(\beta), \tag{14}$$

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$$S_0(\beta) = -\frac{1}{2} + \frac{4}{9}\beta - \frac{4}{27}\beta^2 + \frac{32}{729}\beta^3 - \frac{176}{6561}\beta^4 + \dots, \tag{15}$$

$$\beta = 9\alpha/4\,V_0\,. \tag{16}$$

The condition for the series (15) to converge is

$$0 \leqq \beta \leqq 1\,.$$

For several values of β the corresponding E_0 values are determined. Taking $V_0 = 14.33\,g$ $(0.1 \leqq g \leqq 1.5)$ as given in [5], the ground state energy for deuteron [6], $|E_0| = 2.2245$ MeV, is calculated.

First Excited State

The techniques are similar to the $l=0$ case except that now the wave function of the first state is written as

$$u_1(r) = (x - c)\,e^{\Phi_1(x)}. \tag{17}$$

When $u_1(r)$ is substituted in (3) an equation similar to (7) is obtained, here E_1 is the energy of the first excited state. Power series for $\Phi_1'(x)$ and $\mathscr{E}_1$ of the form of (11) and (12) and

$$c = \sum_{n=-1}^{\infty} C^{(n)}\,k^{-n} \tag{18}$$

are substituted, and equating terms of the same order in k again, the recurrence relations are obtained. The equations are solved for $\Phi_1^{(n)}$, $C^{(n)}$ and $E_1^{(n)}$ in a similar manner as has been done for the groud state.

The energy of the excited state in units of MeV is

$$E_1 = \frac{V_0^2}{\varepsilon}\,S_1(\beta), \tag{19}$$

where

$$S_1(\beta) = -\frac{1}{8} + \frac{4}{9}\beta - \frac{16}{27}\beta^2 + \frac{448}{729}\beta^3 - \dots\,. \tag{20}$$

On chosing three values for β and using the V_0 values already used for the ground state, the following first excited state energies are obtained:

$\beta = 0.0$ and $V_0 = 13.4465$ MeV: $E_1 = -0.5561$ MeV,
$\beta = 0.2$ and $V_0 = 14.7288$ MeV: $E_1 = -0.2930$ MeV,
$\beta = 0.4$ and $V_0 = 16.2159$ MeV: $E_1 = -0.0175$ MeV.

When $\beta > 0.4$, E_1 changes sign, which implies that the bound state does not longer exist.

We conclude that by using the large-N expansion we obtained results for the ground and first excited states of the deuteron that are consistent with the results calculated with other methods, [5], p. 268. The Yukawa type interaction can easily be transformed into a Coulomb type interaction in the limit $\alpha \to 0$. The contribution of the charge interactions to the energies may also be calculated in a similar way using $1/N$ expansions. It is also possible to obtain solutions by choosing more realistic nucleon-nucleon potentials [7].

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